# **Exercise Sheet 8** due 11 December 2014

## 1. *Legendre polynomials*

The Legendre polynomials $L_n(x)$ are defined by the recursion relation

$$(n+1)L_{n+1}(x) = (2n+1)xL_n(x) - nL_{n-1}(x) \tag{1}$$

with $L_0(x) = 1$ and $L_1(x) = x$.

i. Write a program to calculate a symbolic representation of the Legendre polynomials, plot the $L_n(x)$ on the range $x \in [-1, 1]$, and verify that the $L_n$ are orthogonal on the interval $[-1, 1]$: $\int_{-1}^{1} L_n(x)\, L_m(x)\, dx = \frac{2}{2n+1}\delta_{n,m}$.

ii. Show (by induction) that the Legendre polynomials are the solutions to the eigenvalue problem

$$-\left(\frac{d}{dx}\,(1-x^2)\frac{d}{dx}\right) L_n(x) = n(n+1)\, L_n(x)\ . \tag{2}$$

Show that with the substitution $x = \cos(\vartheta)$ this becomes the eigenvalue equation for $\vec{L}^2$ for $m = 0$. Write the spherical harmonics $Y_{l,m=0}(\vartheta, \varphi)$ in terms of the Legendre polynomials.

## 2. *spin*

Compare the commutators of the Pauli matrices of last week's exercises with the commutation relation of the components of the orbital angular momentum $\hat{L}_x$, $\hat{L}_y$, and $\hat{L}_z$. Relate the eigenvalues of $\vec{\sigma}^2$ and $\hat{\sigma}_z$ to what we derived for general angular-momentum operators.

## 3. *Angular momentum operator in spherical coordinates*

Given

$$\begin{aligned}
\hat{L}_x &= i\hbar\left(+\sin\phi\,\frac{\partial}{\partial\theta} + \cot\theta\cos\phi\,\frac{\partial}{\partial\phi}\right)\\
\hat{L}_y &= i\hbar\left(-\cos\phi\,\frac{\partial}{\partial\theta} + \cot\theta\sin\phi\,\frac{\partial}{\partial\phi}\right)\\
\hat{L}_z &= -i\hbar\,\frac{\partial}{\partial\phi}
\end{aligned}$$

i. Show that

$$\hat{L}_\pm = \hat{L}_x \pm i\hat{L}_y = \hbar\, e^{\pm i\phi}\left(\pm\frac{\partial}{\partial\theta} + i\cot\theta\,\frac{\partial}{\partial\phi}\right)$$

ii. Show that

$$\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2 = -\hbar^2\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)$$

Hint: $\hat{L}^2 = \hat{L}_+\hat{L}_- + \hat{L}_z(\hat{L}_z - \hbar)$